

# COHERENT ORIENTATIONS IN SYMPLECTIC FIELD THEORY

FRÉDÉRIC BOURGEOIS AND KLAUS MOHNKE

**ABSTRACT.** We study the coherent orientations of the moduli spaces of holomorphic curves in Symplectic Field Theory, generalizing a construction due to Floer and Hofer. In particular we examine their behavior at multiple closed Reeb orbits under change of the asymptotic direction. The orientations are determined by a certain choice of orientation at each closed Reeb orbit, that is similar to the orientation of the unstable tangent spaces of critical points in finite-dimensional Morse theory.

## 1. INTRODUCTION AND MAIN RESULTS

Contact homology and Symplectic Field Theory were introduced by Eliashberg, Givental and Hofer (see [1] and [2]). They can be considered as a generalization of Floer homology and Gromov–Witten invariants to obtain invariants for contact manifolds via the study of holomorphic curves in symplectizations. Our exposition is a contribution to the foundation of these theories. We construct a gluing consistent orientation of the moduli spaces of punctured holomorphic curves in so-called symplectic cobordisms. The latter are open symplectic manifolds  $(W^{2n}, \omega)$  of dimension  $2n$  such that outside a compact set there are nowhere vanishing, symplectically conformal, complete vector fields. In other words, the ends are symplectomorphic to either  $([R, \infty) \times M_+, d(e^t \alpha_+))$  or  $((-\infty, R] \times M_-, d(e^t \alpha_-))$ , i.e. the positive or negative half of the symplectization of contact manifolds  $(M_\pm, \xi_\pm)$  with  $\xi_\pm := \ker(\alpha_\pm)$ . The symplectic cobordism is equipped with an almost complex structure  $J$  which is compatible to the symplectic structure  $\omega$  and to the contact 1-forms on the ends. This means that  $\omega(\cdot, J\cdot)$  gives a Riemannian structure on  $W$ , that near each end the almost complex structure  $J$  is translationally invariant,  $J(\frac{\partial}{\partial t}) = R_\alpha$  (where  $R_\alpha$  is the Reeb vector field defined by  $i_{R_\alpha} d\alpha = 0, \alpha(R_\alpha) = 1$ ), and  $J$  preserves  $\xi$ .

We construct orientations on the moduli spaces of holomorphic curves, satisfying some relations under the gluing operation. An outline of such a construction was given in [2], but uses a different approach than ours.

Let us assume that the flow of the Reeb vector field  $R_\alpha$  at each end of  $M$  is non-degenerate, i.e. for all its closed orbits (including the multiples) their Poincaré return maps on  $\xi$  have no eigenvalue 1. We denote by  $\mathcal{P}_\alpha$  the set of all closed orbits of the Reeb

flow, where we consider multiples of an orbit as different elements. For each orbit  $\gamma \in \mathcal{P}_\alpha$ , we fix a trivialization of  $\xi$  along  $\gamma$ . Then the linearized Reeb flow on  $\xi$  along  $\gamma$  defines a path in the symplectic group  $Symp(2n-2; \mathbb{R})$ , starting at the identity and ending at a matrix with all eigenvalues different from 1. The Conley–Zehnder index,  $\mu(\gamma) \in \mathbb{Z}$ , is the Maslov index of this path (see [8]).

Let us consider a disk  $D \subset \mathbb{C}$  as a complex curve with a puncture at the origin 0 and a fixed direction at the puncture, namely  $\theta = 0$  where  $(\rho, \theta)$  are the standard polar coordinates of  $\mathbb{C}$ . We say that a smooth map  $u : D \setminus \{0\} \rightarrow \mathbb{R} \times M$  is asymptotic to  $\gamma$  at  $\pm\infty$  if  $u(\rho, \theta) = (u_\mathbb{R}(\rho, \theta), u_M(\rho, \theta))$ ,  $\lim_{\rho \rightarrow 0} u_\mathbb{R}(\rho, \theta) = \pm\infty$  and the uniform limit  $\lim_{\rho \rightarrow 0} u_M(\rho, \theta)$  exists and parameterizes  $\gamma$  in such a way that  $\lim_{\rho \rightarrow 0} u_M(\rho, 0) = z_\gamma$ . We thus may think of the additionally introduced direction at the puncture as a puncture on the boundary of the compactification  $[0, 1] \times S^1$  of  $D \setminus \{0\}$  via polar coordinates. The map  $u$  thus extends to the compactification and maps the puncture on the boundary to  $z_\gamma$ .

Let  $(\Sigma, j)$  be a Riemann surface,  $j$  its conformal structure. Let  $\bar{x}_1, \dots, \bar{x}_s; \underline{x}_1, \dots, \underline{x}_s \in \Sigma$  be distinct points of the surface (which we will call *punctures*) and  $\bar{v}_1 \in T_{\bar{x}_1}\Sigma, \dots, \bar{v}_s \in T_{\bar{x}_s}\Sigma; \underline{v}_1 \in T_{\underline{x}_1}\Sigma, \dots, \underline{v}_s \in T_{\underline{x}_s}\Sigma$  be fixed unit vectors in the tangent spaces at the points, which we call *directions*. We say that a map  $u : \Sigma \setminus \{\bar{x}_1, \dots, \bar{x}_s, \underline{x}_1, \dots, \underline{x}_s\} \rightarrow W$  is asymptotic to  $\bar{\gamma}_k \in \mathcal{P}_{\alpha_+}$  in  $\bar{x}_k$  at  $+\infty$  if there are complex polar coordinates  $(\rho, \theta)$  centered at  $\bar{x}_k$  such that the directions  $\bar{v}_k$  coincide with  $\theta = 0$  and  $u$  is asymptotic to  $\bar{\gamma}_k$  at  $+\infty$ . Analogously, we say that  $u$  is asymptotic to  $\underline{\gamma}_l \in \mathcal{P}_{\alpha_-}$  in  $\underline{x}_l$  at  $-\infty$ .

Note that the diffeomorphisms of  $\Sigma$  act in an obvious way on the data described above, giving rise to new conformal structures and new holomorphic maps with the same asymptotics. We are now ready to define the moduli spaces of holomorphic curves in a symplectic cobordism  $(W, J)$ .

Pick a closed Riemann surface  $\Sigma$ , and closed orbits  $\bar{\gamma}_1, \dots, \bar{\gamma}_s \in \mathcal{P}_{\alpha_+}$  and  $\underline{\gamma}_1, \dots, \underline{\gamma}_s \in \mathcal{P}_{\alpha_-}$  of the flow of the corresponding Reeb vector fields. Define the moduli space

$$\mathcal{M}_{W,J}^\Sigma(\bar{\gamma}_1, \dots, \bar{\gamma}_s; \underline{\gamma}_1, \dots, \underline{\gamma}_s)$$

to be the set of conformal structures on  $\Sigma$  together with  $\bar{s}$  positive and  $\underline{s}$  negative punctures with directions, and holomorphic maps into  $(W, J)$  asymptotic to  $\bar{\gamma}_1, \dots, \bar{\gamma}_s$  at  $+\infty$  at the positive punctures and to  $\underline{\gamma}_1, \dots, \underline{\gamma}_s$  at  $-\infty$  at the negative punctures, modulo the action by the diffeomorphisms of  $\Sigma$ .

Two symplectic cobordisms  $(W_1, \omega_1)$  and  $(W_2, \omega_2)$  can be glued if the negative end of  $(W_1, \omega_1)$  has the form  $((-\infty, -R_0] \times M, d(e^t\alpha))$  and the positive end of  $(W_2, \omega_2)$  looks like  $([+R_0, +\infty) \times M, d(e^t\alpha))$ . The glued cobordism  $W_R$ , for  $R > R_0$ , is obtained after cutting  $W_1$  along  $\{-R\} \times M$ ,  $W_2$  along  $\{+R\} \times M$  and identifying the ends with the

identity map. We obtain the glued symplectic structure  $\omega_R$  after multiplying  $\omega_1$  by  $e^R$  and  $\omega_2$  by  $e^{-R}$ . If the  $(W_i, \omega_i)$  are equipped with compatible almost complex structures that are identical on the ends we glue, then the glued cobordism  $(W_R, \omega_R)$  is naturally equipped with a compatible almost complex structure  $J_R$ . Of course, if at least one of the  $(W_i, \omega_i)$  is a symplectization equipped with an  $\mathbb{R}$ -invariant almost complex structure, the glued structures  $(W_R, \omega_R, J_R)$  are independent of  $R > R_0$ .

We can also glue holomorphic curves in the almost complex manifolds  $(W_i, J_i)$ . Consider compact subsets

$$K_1 \subset \mathcal{M}_{W_1, J_1}^{\Sigma_1}(\overline{\gamma}_1, \dots, \overline{\gamma}_{\underline{s}}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}-t}, \beta_1, \dots, \beta_t)$$

and

$$K_2 \subset \mathcal{M}_{W_2, J_2}^{\Sigma_2}(\beta_1, \dots, \beta_t, \overline{\gamma}'_{t+1}, \dots, \overline{\gamma}'_{\underline{s}'}; \underline{\gamma}'_1, \dots, \underline{\gamma}'_{\underline{s}'})$$

of a pair of moduli spaces. Let  $\Sigma = \Sigma_1 \#_t \Sigma_2$  be the Riemann surface obtained by gluing  $\Sigma_1$  and  $\Sigma_2$  at the punctures with asymptotics  $\beta_1, \dots, \beta_t$ , identifying their asymptotic directions. If  $(W_2, \omega_2, J_2)$  is a symplectization equipped with an  $\mathbb{R}$ -invariant almost complex structure, we can have  $\underline{s} > t$  since a holomorphic curve in  $(W_1, J_1)$  can be extended in  $(W_2, J_2)$  by a cylinder over a closed Reeb orbit; otherwise, we must have  $\underline{s} = t$ . There is a similar remark for  $(W_1, \omega_1, J_1)$ .

One constructs, for  $R$  large enough, a diffeomorphism

$$\Phi_R : K_1 \times K_2 \longrightarrow \mathcal{M}_{W_R, J_R}^{\Sigma}(\overline{\gamma}_1, \dots, \overline{\gamma}_{\underline{s}}, \overline{\gamma}'_{t+1}, \dots, \overline{\gamma}'_{\underline{s}'}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}-t}, \underline{\gamma}'_1, \dots, \underline{\gamma}'_{\underline{s}'}).$$

In this paper, we will construct and use a linearized version of this *gluing diffeomorphism*.

Each moduli space  $\mathcal{M}_{W, J}^{\Sigma}$  comes naturally equipped with a real line bundle, called determinant bundle. Whenever the moduli spaces are cut out transversally by the Cauchy-Riemann equation (or a small perturbation of these equations), the determinant bundle is canonically isomorphic to the orientation bundle  $\Lambda^{\max} T\mathcal{M}_{W, J}^{\Sigma}$  of  $\mathcal{M}_{W, J}^{\Sigma}$ . In particular, an orientation of the determinant bundle of  $\mathcal{M}_{W, J}^{\Sigma}$  is equivalent to an orientation of  $\mathcal{M}_{W, J}^{\Sigma}$ . Using gluing analysis of Fredholm operators, similar to that of Floer and Hofer in [3], we will construct orientations for the determinant bundles of the moduli spaces :

**Theorem 1.** *The determinant bundles of the moduli spaces  $\mathcal{M}_{W, J}^{\Sigma}$  are orientable in such a way that the gluing diffeomorphisms preserve the orientations (up to a sign due to the reordering of some punctures, see Corollary 10) and the orientation of a disjoint union of data is induced by the orientation on each component (see Proposition 9).*

In this article we describe the behavior of the constructed orientations under the change of certain data. The first property is a relation for the orientations of the moduli spaces

when reordering the punctures. This has been known since the early days of Symplectic Field Theory, for algebraic reasons : it is necessary for the differential  $d$  to be well-defined.

**Theorem 2.** *Let*

$$\begin{aligned} \underline{l}, \underline{l}+1 : \mathcal{M}_{W,J}^{\Sigma}(\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{s}}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{l}}, \underline{\gamma}_{\underline{l}+1}, \dots, \underline{\gamma}_{\underline{s}}) \\ \longrightarrow \mathcal{M}_{W,J}^{\Sigma}(\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{s}}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{l}+1}, \underline{\gamma}_{\underline{l}}, \dots, \underline{\gamma}_{\underline{s}}) \end{aligned}$$

*be the map induced by exchanging the  $l$ th and the  $(l+1)$ th negative punctures. There is a natural identification of the determinant bundles of the corresponding linearizations for  $[u, j]$  and  $\underline{l}, \underline{l}+1([u, j])$ , which are simply the same. Then the orientations given in Proposition 1*

- (i) *agree, if the product  $(\mu(\underline{\gamma}_{\underline{l}}) + (n-3))(\mu(\underline{\gamma}_{\underline{l}+1}) + (n-3))$  is even;*
- (ii) *disagree, if it is odd.*

*A similar statement holds for the change of the ordering at the positive end.*

The contribution of the paper is to explain this by the consistency relation under disjoint union. Next we study the behavior of the orientation under the change of asymptotic directions at the punctures which are asymptotic to multiply covered closed Reeb orbits. We say that a closed Reeb orbit  $\gamma_m$  is *bad* if it is the  $m$ -fold covering of some Reeb orbit  $\gamma$  and the difference  $\mu(\gamma_m) - \mu(\gamma)$  of their Conley–Zehnder indices is odd. See e.g. [10] for a discussion. If this happens, then the integer  $m$  must be even. Closed Reeb orbits which are not bad are called *good*.

**Theorem 3.** *Consider the moduli space  $\mathcal{M}^{\Sigma}(\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{s}}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}})$ , where  $\bar{\gamma}_k$  is the  $m$ -fold covering of a closed Reeb orbit. There is a natural  $\mathbb{Z}_m$ -action on it which acts transitively on the possible direction at  $\bar{x}_k$ . This action is orientation preserving if and only if  $\bar{\gamma}_k$  is good. A similar statement holds in the case of a negative puncture.*

At this point, the role of the chosen points  $z_{\gamma}$  and of the asymptotic directions becomes clear. Otherwise, it would be impossible to fix an orientation for holomorphic curves with at least one puncture asymptotic to a multiple Reeb orbit with odd behavior. The necessity of such a result has been clear since Michael Hutchings observed that in order to have invariance of Symplectic Field Theory, certain closed Reeb orbits should be ignored.

We will discuss the Fredholm theory for linearized Cauchy-Riemann operators, corresponding to holomorphic curves in symplectic cobordisms, in Section 2. Then in Section 3 we will generalize the construction of coherent orientations of Floer and Hofer. In Section 4, we apply it to the moduli spaces of holomorphic curves in Symplectic Field Theory. Finally, in Section 5, we will discuss the even and odd behavior under the change of asymptotic directions at multiple closed Reeb orbits described in Theorem 3.

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## 2. FREDHOLM THEORY

Let  $\Sigma$  be a closed Riemann surface with conformal structure  $j$ , positive punctures  $\bar{x}_k$ ,  $k = 1, \dots, \bar{s}$  and negative punctures  $\underline{x}_l$ ,  $l = 1, \dots, \underline{s}$ . On a small disk of radius  $e^{-R_0}$  centered on a puncture  $\bar{x}_k$  or  $\underline{x}_l$ , we will use a local complex coordinate  $z = e^{\mp(s+i\theta)}$  vanishing at  $\bar{x}_k$  or  $\underline{x}_l$ . Then  $(s, \theta) \in [R_0, \infty) \times S^1$  for a positive puncture and  $(s, \theta) \in (-\infty, -R_0] \times S^1$  for a negative puncture;  $(s, \theta)$  are cylindrical coordinates on the punctured disk. We obtain a compactification  $\bar{\Sigma}$  of the Riemann surface  $\Sigma$  by extending the coordinate  $s$  to  $\pm\infty$  at each puncture.

Let  $E$  be a symplectic vector bundle over the closed Riemann surface  $\Sigma$  together with symplectic linear identifications  $E_{\bar{x}_k} \cong E_{\underline{x}_l} \cong (\mathbb{R}^2, \omega_0) \oplus (\mathbb{R}^{2n-2}, \omega_0)$ . This gives rise to a symplectic vector bundle  $\bar{E}$  over the compactification  $\bar{\Sigma}$  with fixed trivializations on the boundary components.

Let  $\beta : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\beta(s) = 0$  if  $s < 0$ ,  $\beta(s) = 1$  if  $s > 1$  and  $0 \leq \beta'(s) \leq 2$ . Let  $\mathfrak{e}_{k,a}$ ,  $a = 1, 2$  be vectors spanning the first summand of  $E_{\bar{x}_k} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$ . We similarly define vectors  $\underline{e}_{l,a}$ . We define  $\Gamma_{\bar{s}, \underline{s}} \cong \mathbb{R}^{2\bar{s}+2\underline{s}}$  to be the vector space generated by the sections  $\bar{w}_{k,a}(s) = \beta(|s| - R_0)\mathfrak{e}_{k,a}$  and  $\underline{w}_{l,a}(s) = \beta(|s| - R_0)\underline{e}_{l,a}$ .

Let  $L_k^{p,d}(E)$  be the Sobolev space of sections  $\zeta$  of  $E$  that are locally in  $L_k^p$  and such that, near each puncture,  $e^{\frac{d}{p}|s|}\zeta(s, \theta)$  is in  $L_k^p$  with respect to the cylindrical measure  $ds d\theta$ . The Banach norm on  $L_k^{p,d}(E)$  is defined with respect to the measure  $e^{d|s|}ds d\theta$  near the punctures.

Let

$$\mathcal{S} := \{ A : [0, 2\pi] \longrightarrow \text{Symp}(2n-2; \mathbb{R}) \mid 1 \notin \text{spec}(A(2\pi)); \\ A(0) = \text{Id}, \dot{A}(0)A(0)^{-1} = \dot{A}(2\pi)A(2\pi)^{-1} \}$$

be the set of regular paths in the symplectic group. We will denote the Maslov index of a path (see [8]) of symplectic matrices  $A \in \mathcal{S}$  by  $\mu(A)$ . Note that for a regular closed Reeb orbit  $\gamma$  with a fixed symplectic trivialization of  $\xi$ , the linearization of the Reeb flow along  $\gamma$  determines an element  $A_\gamma \in \mathcal{S}$ .

Now, for  $\overline{A}_k, \underline{A}_l \in \mathcal{S}$  let  $\mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}})$  be the set of continuous linear operators  $L : \Gamma_{\overline{s}, \underline{s}} \oplus L_k^{p,d}(E) \longrightarrow L_{k-1}^{p,d}(\Lambda^{0,1}(E))$ , with  $p > 2$ ,  $k \geq 1$  and  $d > 0$ , of the following form :

- (i) in a trivialization of  $E$  in the interior of  $\Sigma$ , with local coordinate  $z = x + iy$ , the operator  $L$  looks like

$$\left( \frac{\partial}{\partial x} + J(z) \frac{\partial}{\partial y} + S(z) \right) (dx - idy),$$

where  $J(z)$  and  $S(z)$  are locally in  $L_k^p$  and  $J(z)$  is a compatible complex structure on  $E$ ;

- (ii) in a trivialization of  $\overline{E}$  near a puncture, that extends the fixed identification at the puncture, with the cylindrical coordinates  $(s, \theta)$ , the operator  $L$  looks like

$$\left( \frac{\partial}{\partial s} + J(s, \theta) \frac{\partial}{\partial \theta} + S(s, \theta) \right) (ds - id\theta),$$

where  $S(s, \theta)$  and  $J(s, \theta)$  extend continuously to the compactification  $\overline{\Sigma}$  in such a way that  $J(s, \theta) - J(\pm\infty, \theta)$  and  $S(s, \theta) - S(\pm\infty, \theta)$  are in  $L_{k-1}^{p,d}$ , and  $J(s, \theta)$  is a compatible complex structure on  $\overline{E}$ .

The loop of matrices  $S(+\infty, \theta)$  splits in  $E_{\overline{x}_k} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$  as

$$S(+\infty, \theta) = \begin{pmatrix} 0 & 0 \\ 0 & \overline{S}_k(\theta) \end{pmatrix},$$

where

$$\overline{S}_k(\theta) := -J(+\infty, \theta) \dot{\overline{A}}_k(\theta) \overline{A}_k(\theta)^{-1}$$

is a loop of  $(2n-2) \times (2n-2)$ -matrices which are symmetric with respect to the euclidean structure determined by the symplectic and the complex structure. There is a similar expression for the loop of matrices  $S(-\infty, \theta)$ .

Note that the differential operators we consider have degenerate asymptotics, because the matrices  $S$  vanish on the first summand of  $E_{\overline{x}_k} \cong E_{\underline{x}_l} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$ . Hence, we cannot directly apply to them the usual Fredholm theory as in [9].

**Proposition 4.** *Let  $L \in \mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}})$ . Then*

$$L : \Gamma_{\overline{s}, \underline{s}} \oplus L_k^{p,d}(E) \longrightarrow L_{k-1}^{p,d}(\Lambda^{0,1}(E))$$

*is a Fredholm operator with index*

$$\text{ind}(L) = \sum_{i=1}^{\overline{s}} (\mu(\overline{A}_i) - (n-1)) - \sum_{j=1}^{\underline{s}} (\mu(\underline{A}_j) + (n-1)) + n\chi(\Sigma) + 2c_1(E).$$

$\chi(\Sigma)$  denotes the Euler characteristic of  $\Sigma$ .

*Proof.* First consider the restriction of  $L$  to  $L_k^{p,d}(E)$ . Note that the  $L_k^{p,d}$  spaces are isomorphic to the  $L_k^p$  spaces, via multiplication by a function that is constant away from the punctures and given by  $e^{\frac{d}{p}|s|}$  near the punctures. Conjugating our restricted operator with these isomorphisms, we obtain an operator  $L'$  from  $L_k^p$  to  $L_{k-1}^p$ . Moreover, near a puncture  $\bar{x}_k$  or  $\underline{x}_l$ , the operator  $L'$  looks like

$$\left( \frac{\partial}{\partial s} + J(s, \theta) \frac{\partial}{\partial \theta} + S(s, \theta) \mp \frac{d}{p} I \right) (ds - i d\theta).$$

In particular, if the operator  $L$  has asymptotics  $\bar{A}_k, \underline{A}_l \in \mathcal{S}$ , the new operator  $L'$  has asymptotics  $\bar{A}'_k, \underline{A}'_l$ , where

$$\bar{A}'_k(\theta) = e^{-i\frac{d}{p}\theta} \begin{pmatrix} I & 0 \\ 0 & \bar{A}_k(\theta) \end{pmatrix} \quad \text{and} \quad \underline{A}'_l(\theta) = e^{+i\frac{d}{p}\theta} \begin{pmatrix} I & 0 \\ 0 & \underline{A}_l(\theta) \end{pmatrix}.$$

As a consequence of this small perturbation, the asymptotics are not degenerate anymore, and we obtain a Fredholm operator (see [9]). Its index is given by

$$n(2 - 2g - \bar{s} - \underline{s}) + 2c_1(E) + \sum_{k=1}^{\bar{s}} \mu(\bar{A}'_k) - \sum_{l=1}^{\underline{s}} \mu(\underline{A}'_l).$$

It is easy to see that  $\mu(\bar{A}'_k) = \mu(\bar{A}_k) - 1$  and  $\mu(\underline{A}'_l) = \mu(\underline{A}_l) + 1$ .

Let us now consider the summand  $\Gamma_{\bar{s}, \underline{s}}$ . The images under  $L$  of these sections have their support near a puncture and decay exponentially near that puncture, hence  $L$  maps the new elements into  $L_{k-1}^{p,d}$ . The space  $\Gamma_{\bar{s}, \underline{s}}$  is finite dimensional. Thus the operator  $L$  is Fredholm. Its Fredholm index is the sum of the index of the operator restricted to  $L_k^{p,d}(E)$  and the dimension of the vector space  $\Gamma_{\bar{s}, \underline{s}}$ , which is  $2(\bar{s} + \underline{s})$ . Hence we obtain the desired formula.  $\square$

As in [3], the sets  $\mathcal{O}(\Sigma, E; \bar{A}_1, \dots, \bar{A}_{\bar{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}})$  are topological spaces. Moreover, as in Proposition 8 of [3], these spaces are contractible. Note that unlike in [3] we allow the complex structures to vary on the ends here. Since the space  $\mathcal{J}(\bar{E})$  of compatible complex structures is contractible this does not cause any trouble similar to that of Theorem 2 of [3]. The advantage will be that we can make orientation choices for each closed Reeb orbit together with its fixed trivialization of  $\xi$  without further structure.

By Proposition 4, the topological spaces  $\mathcal{O}(\Sigma, E; \bar{A}_1, \dots, \bar{A}_{\bar{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}})$  consist of Fredholm operators. Recall that, for every Fredholm operator  $L$ , the determinant space of  $L$  is defined by  $\text{Det}(L) := (\Lambda^{\max} \ker L) \otimes (\Lambda^{\max} \text{coker } L)^*$ . Moreover, the determinant spaces of  $L \in \mathcal{O}$  fit together in a determinant bundle  $\text{Det}(\mathcal{O})$  over  $\mathcal{O}$ . Since the spaces  $\mathcal{O}$  are contractible, this determinant bundle is orientable. In order to orient these determinant

bundles in a coherent fashion, we need to construct a gluing map to relate orientations on different bundles  $\text{Det}(\mathcal{O})$ .

Consider two operators

$$\begin{aligned} K &\in \mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\underline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}}), \\ L &\in \mathcal{O}(\Sigma', E'; \overline{A}'_1, \dots, \overline{A}'_{\underline{s}'}; \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'}), \end{aligned}$$

such that  $\underline{A}_{\underline{s}+1-m}$  coincides with  $\overline{A}'_m$  for  $m = 1, \dots, t \leq \min(\underline{s}, \underline{s}')$ . We want to construct a glued operator

$$M_R \in \mathcal{O}(\Sigma'', E''; \overline{A}_1, \dots, \overline{A}_{\underline{s}}, \overline{A}'_{t+1}, \dots, \overline{A}'_{\underline{s}'}; \underline{A}_1, \dots, \underline{A}_{\underline{s}-t}, \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'}).$$

Near the punctures  $\underline{x}_{\underline{s}+1-m}$ , pick cylindrical coordinates  $(s_m, \theta_m) \in (-\infty, -R_0] \times S^1$ ; near the punctures  $\overline{x}'_m$ , pick cylindrical coordinates  $(s'_m, \theta'_m) \in [R_0, +\infty) \times S^1$ . Choose  $R$  much larger than  $R_0$ . Cut out the small disks  $\{s_m < -R\}$  from  $\Sigma$  and  $\{s'_m > +R\}$  from  $\Sigma'$  and identify their boundaries for  $m = 1, \dots, t$  so that  $\theta_m = \theta'_m$ . We denote the resulting Riemann surface by  $\Sigma''$ . Gluing the bundles  $E$  and  $E'$  using their trivializations near the punctures, we obtain a bundle  $E''$  over  $\Sigma''$ . In the neighborhood of the gluing, we obtain cylindrical coordinates  $(s''_m, \theta''_m)$  defined by  $s''_m = s_m + R$ ,  $\theta''_m = \theta_m$  on  $\Sigma$  and  $s''_m = s'_m - R$ ,  $\theta''_m = \theta'_m$  on  $\Sigma'$ .

In cylindrical coordinates  $(s''_m, \theta''_m)$ , we define

$$M_R := \left( \frac{\partial}{\partial s''_m} + J''(s''_m, \theta''_m) \frac{\partial}{\partial \theta''_m} + S''(s''_m, \theta''_m) \right) (ds''_m - \text{id} \theta''_m),$$

where

$$\begin{aligned} S''(s''_m, \theta''_m) &= S(-\infty, \theta''_m) + \beta(s''_m)(S(s_m, \theta_m) - S(-\infty, \theta_m)) \\ &\quad + \beta(-s''_m)(S'(s'_m, \theta'_m) - S'(+\infty, \theta'_m)) \end{aligned}$$

and  $J''(s''_m, \theta''_m)$ ,  $s''_m \in [-1, +1]$ , is a path of compatible complex structures on  $E''$  interpolating between  $J(-R+1, \theta_m)$  and  $J'(R-1, \theta'_m)$ .

Note that, on  $L_k^{p,d}(E'')$ , we will not use the usual Banach norm, but a modified norm. For  $\zeta \in L_k^{p,d}(E'')$ , let  $\bar{\zeta}_m = \frac{1}{2\pi} \int_{s''_m=0} \pi_m \zeta d\theta''_m$ , where  $\pi_m$  is the projection to the first summand of  $E_{\underline{x}_{\underline{s}-t+m}} \cong E_{\overline{x}'_m} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$ . Let  $\bar{\beta}(s) = \beta(-s + R - R_0)\beta(s + R - R_0)$ . We define

$$\|\zeta\|_{L_k^{p,d}(E'')} = \|\zeta - \sum_{m=1}^t \bar{\beta}(s''_m) \bar{\zeta}_m\|_R + \sum_{m=1}^t |\bar{\zeta}_m|,$$

where  $\|\cdot\|_R$  is the Banach norm with respect to a measure of the same form as before near the punctures but with a measure  $e^{d(R-R_0-|s''_m|)} ds''_m d\theta''_m$  in the neighborhood of the gluing.



On the other hand, on  $L_{k-1}^{p,d}(\Lambda^{0,1}(E''))$ , we will use the Banach norm  $\|\cdot\|_R$ , with a measure of the same form as before near the punctures but with a measure  $e^{d(R-R_0-|s_m''|)} ds_m'' d\theta_m''$  in the neighborhood of the gluing.

Let  $\Gamma_t$  be the vector space generated by the pairs  $(\underline{w}_{\underline{s}+1-m,a}, \overline{w}'_{m,a}) \in \Gamma_{\overline{s},\underline{s}} \oplus \Gamma'_{\overline{s}',\underline{s}'}$ , for  $m = 1, \dots, t$  and  $a = 1, 2$ . We write  $\overline{s}'' = \overline{s} + \overline{s}' - t$  and  $\underline{s}'' = \underline{s} + \underline{s}' - t$ . Let  $\Gamma''_{\overline{s}'',\underline{s}''}$  be the vector space generated by the sections  $\overline{w}_{k,a}$ , for  $k = 1, \dots, \overline{s}$ , the sections  $\underline{w}_{l,a}$  for  $l = 1, \dots, \underline{s} - t$ , the sections  $\overline{w}'_{k,a}$ , for  $k = t+1, \dots, \overline{s}'$ , and the sections  $\underline{w}'_{l,a}$ , for  $l = 1, \dots, \underline{s}'$  and  $a = 1, 2$ . We denote by  $K \oplus_{\Gamma_t} L$  the restriction of  $K \oplus L$  to  $\Gamma''_{\overline{s}'',\underline{s}''} \oplus \Gamma_t \oplus L_k^{p,d}(E) \oplus L_k^{p,d}(E')$ .

**Proposition 5.** *Assume that the operator  $K \oplus_{\Gamma_t} L$  is surjective. Then the operator*

$$M_R : \Gamma''_{\overline{s}'',\underline{s}''} \oplus L_k^{p,d}(E'') \rightarrow L_{k-1}^{p,d}(\Lambda^{0,1}(E''))$$

*has a uniformly bounded right inverse  $Q_R$ , if  $R$  is sufficiently large.*

In order to prove this proposition, we adapt the gluing construction of McDuff and Salamon [7].

Let  $\gamma_R : \mathbb{R} \rightarrow [0, 1]$  be a smooth decreasing function such that  $\gamma_R(s) = 0$  for  $s \geq \frac{R-R_0}{2}$ ,  $\gamma_R(s) = 1$  for  $s \leq -1$  and all derivatives of  $\gamma_R$  uniformly converge to 0 as  $R \rightarrow \infty$ .

Let us define the gluing map

$$\begin{aligned} g_R : \Gamma_t \oplus L_k^{p,d}(E) \oplus L_k^{p,d}(E') &\rightarrow L_k^{p,d}(E'') \\ (v, \zeta, \zeta') &\rightarrow \zeta'' = \zeta^0 + v \end{aligned}$$

where, in the neck with coordinates  $(s_m'', \theta_m'')$ ,

$$\zeta^0(s_m'', \theta_m'') = \begin{cases} \zeta(s_m'', \theta_m'') + \gamma_R(s_m'') \zeta'(s_m'', \theta_m'') & \text{if } s_m'' > +1 \\ \zeta(s_m'', \theta_m'') + \zeta'(s_m'', \theta_m'') & \text{if } -1 \leq s_m'' \leq +1 \\ \gamma_R(-s_m'') \zeta(s_m'', \theta_m'') + \zeta'(s_m'', \theta_m'') & \text{if } s_m'' < -1 \end{cases}$$

and  $\zeta^0$  coincides with  $\zeta$  and  $\zeta'$  on the rest of  $\Sigma$  and  $\Sigma'$  respectively.

Let us define the splitting map

$$\begin{aligned} s_R : L_{k-1}^{p,d}(\Lambda^{0,1}(E'')) &\rightarrow L_{k-1}^{p,d}(\Lambda^{0,1}(E)) \oplus L_{k-1}^{p,d}(\Lambda^{0,1}(E')) \\ \eta'' &\rightarrow (\eta, \eta') \end{aligned}$$

where, near the punctures  $\underline{x}_{\underline{s}+1-m}$  and  $\overline{x}'_m$ ,

$$\begin{cases} \eta(s_m, \theta_m) &= \beta(s_m'') \eta''(s_m'', \theta_m'') \\ \eta'(s'_m, \theta'_m) &= (1 - \beta(s_m'')) \eta''(s_m'', \theta_m'') \end{cases}$$

and  $\eta, \eta'$  coincide with  $\eta''$  away from these punctures.

Note that the operators  $g_R$  and  $s_R$  are uniformly bounded in  $R$ .

Let  $Q_\infty$  be a right inverse for the surjective operator  $K \oplus_{\Gamma_t} L$ . Let us define an approximate right inverse  $\tilde{Q}_R$  for  $M$  using the following commutative diagram :

$$\begin{array}{ccc}
L_{k-1}^{p,d}(\Lambda^{0,1}(E'')) & \xrightarrow{\tilde{Q}_R} & \Gamma_{\underline{s}'', \underline{s}''}'' \oplus L_k^{p,d}(E'') \\
s_R \downarrow & & \uparrow g_R \\
L_{k-1}^{p,d}(\Lambda^{0,1}(E)) \oplus L_{k-1}^{p,d}(\Lambda^{0,1}(E')) & \xrightarrow{Q_\infty} & \Gamma_{\underline{s}'', \underline{s}''}'' \oplus \Gamma_t \oplus L_k^{p,d}(E) \oplus L_k^{p,d}(E')
\end{array}$$

Note that  $\tilde{Q}_R$  is uniformly bounded in  $R$ , since  $g_R$  and  $s_R$  are.

*Proof of Proposition 5.* By construction,  $M_R \tilde{Q}_R \eta'' = \eta''$  away from the neighborhoods of the gluing. On the other hand, for  $s_m'' \in [-\frac{R-R_0}{2}, +\frac{R-R_0}{2}]$ , we have

$$\begin{aligned}
M_R \tilde{Q}_R \eta'' &= M_R \zeta'' \\
&= M_R(v_m + \gamma_R(-s_m'')\zeta + \gamma_R(s_m'')\zeta') \\
&= M_R v_m + \gamma_R(-s_m'')K\zeta + \gamma_R(s_m'')L\zeta' - \frac{d}{ds}\gamma_R(-s_m'')\zeta + \frac{d}{ds}\gamma_R(s_m'')\zeta' \\
&\quad + \gamma_R(-s_m'')(M-K)\zeta + \gamma_R(s_m'')(M-L)\zeta'
\end{aligned}$$

But

$$\begin{aligned}
\gamma_R(-s_m'')K\zeta + \gamma_R(s_m'')L\zeta' &= \gamma_R(-s_m'')\eta + \gamma_R(s_m'')\eta' \\
&= \eta + \eta' \\
&= \eta''
\end{aligned}$$

because  $\gamma_R(-s) = 1$  (resp.  $\gamma_R(s) = 1$ ) when  $\eta$  (resp.  $\eta'$ ) is not zero.

Therefore,

$$\begin{aligned}
M_R \tilde{Q}_R \eta'' - \eta'' &= M_R v_m - \gamma_R'(-s_m'')\zeta + \gamma_R'(s_m'')\zeta' \\
&\quad + \gamma_R(-s_m'')(M_R - K)\zeta + \gamma_R(s_m'')(M_R - L)\zeta'.
\end{aligned}$$

Hence, it follows from our constructions that

$$\|M_R \tilde{Q}_R \eta'' - \eta''\|_R \leq C(R)(\|v\| + \|\zeta\| + \|\zeta'\|)$$

where  $C(R) \rightarrow 0$  when  $R \rightarrow \infty$ .

Therefore, if  $R$  is sufficiently large,

$$\|M_R \tilde{Q}_R - I\| \leq \frac{1}{2}.$$

Hence, the operator  $M_R \tilde{Q}_R$  is invertible. Let  $Q_R = \tilde{Q}_R (M_R \tilde{Q}_R)^{-1}$ . By construction,  $Q_R$  is a right inverse for  $M_R$ , and it is uniformly bounded in  $R$ .  $\square$

**Corollary 6.** *There is a natural isomorphism*

$$\phi : \ker K \oplus_{\Gamma_t} \ker L \rightarrow \ker M_R$$

*that is defined up to homotopy, if  $R$  is sufficiently large.*

*Proof.* Let  $Q_R$  be the uniformly bounded right inverse for the glued operator  $M_R \in \mathcal{O}''$ , as in Proposition 5. We define  $\phi$  to be the composition of the map  $g_R$  and the projection map  $(I - Q_R M_R)$  from the domain of  $M_R$  to its kernel, along the image of  $Q_R$ . In other words,  $\phi = (I - Q_R M_R) \circ g_R$ .

We claim that the restriction of  $\phi$  to  $\ker K \oplus_{\Gamma_t} \ker L$  is an isomorphism for  $R$  sufficiently large. By Proposition 4, the dimensions of both spaces agree, so it is enough to show that  $\phi$  is injective.

By contradiction, assume that for any large  $R$ , we can find  $\zeta_R \in \ker K$  and  $\zeta'_R \in \ker L$  such that  $\|\zeta_R\| + \|\zeta'_R\| = 1$  and  $g_R(\zeta_R, \zeta'_R) = Q_R \eta''_R$ , for some  $\eta''_R$ . Since  $K\zeta_R = L\zeta'_R = 0$ , we see that  $\lim_{R \rightarrow \infty} M_R g_R(\zeta_R, \zeta'_R) = 0$  by the same kind of computation as for  $M_R \tilde{Q}_R \eta''$  at the beginning of the proof of Proposition 5. But since  $M_R Q_R = I$ , it follows that  $\eta''_R \rightarrow 0$  when  $R \rightarrow \infty$ . Using this in the original equation gives  $\lim_{R \rightarrow \infty} g_R(\zeta_R, \zeta'_R) = 0$ . But this contradicts  $\|\zeta_R\| + \|\zeta'_R\| = 1$ .  $\square$

Suppose now that the operator  $K \oplus_{\Gamma_t} L$  is not surjective. Then we stabilize the operators  $K$  and  $L$  using finite dimensional oriented vector spaces  $H_1$  and  $H_2$  generated by smooth sections with compact support of  $\Lambda^{0,1}(E)$  and  $\Lambda^{0,1}(E')$  respectively. We obtain operators

$$\begin{aligned} K_{H_1} : H_1 \oplus \Gamma_{\bar{s}, \underline{s}} \oplus L_k^{p,d}(E) &\longrightarrow L_{k-1}^{p,d}(\Lambda^{0,1}(E)) \\ (h_1, v, \zeta) &\longrightarrow h_1 + K(v, \zeta) \\ L_{H_2} : H_2 \oplus \Gamma'_{\bar{s}', \underline{s}'} \oplus L_k^{p,d}(E') &\longrightarrow L_{k-1}^{p,d}(\Lambda^{0,1}(E')) \\ (h_2, v', \zeta') &\longrightarrow h_2 + L(v', \zeta'). \end{aligned}$$

It is a standard property of determinant spaces (see for example [3]) that  $\text{Det}(K)$  and  $\text{Det}(L)$  are canonically isomorphic to  $\text{Det}(K_{H_1})$  and  $\text{Det}(L_{H_2})$ , respectively. We can always choose  $H_1$  and  $H_2$  so that  $K_{H_1} \oplus_{\Gamma_t} L_{H_2}$  is surjective. When  $R$  is large enough, the operator  $K_{H_1} \oplus_{\Gamma_t} L_{H_2}$  is a stabilization of  $K \oplus_{\Gamma_t} L$ ; hence, their determinant spaces are canonically isomorphic.

Moreover, the symplectic orientation of the first summand of  $E_{\underline{x}_{\pm+1-m}} \cong E_{\bar{x}_m} \cong \mathbb{R}^2 \oplus \mathbb{R}^{2n-2}$  induces an orientation on the vector space  $\Gamma_t$ . This in turn induces a natural isomorphism between  $\text{Det}(K \oplus_{\Gamma_t} L)$  and the determinant space of its stabilization  $K \oplus L$ .

Therefore, we can generalize Corollary 6 as follows :

**Corollary 7.** *Given operators  $K$  and  $L$  as above, there is a natural isomorphism*

$$\Psi : \text{Det}(K) \otimes \text{Det}(L) \longrightarrow \text{Det}(M_R)$$

*that is defined up to homotopy.*

### 3. COHERENT ORIENTATIONS OF THE DETERMINANT LINE BUNDLES

We will explain the algorithm to orient all determinant line bundles over the spaces  $\mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}})$  of Fredholm operator. This was originally done in [3] for  $\overline{s} = \underline{s} = 1$ .

In order to construct the orientations, we need the following operations :

(i) Gluing of orientations. If we have elements

$$\begin{aligned} K &\in \mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}}) \\ L &\in \mathcal{O}(\Sigma', E'; \overline{A}'_1, \dots, \overline{A}'_{\overline{s}'}; \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'}) \end{aligned}$$

with  $\underline{A}_{\underline{s}+1-i}$  matching  $\overline{A}'_i$  for  $i = 1, \dots, t \leq \min\{\underline{s}, \overline{s}'\}$  we can glue them to obtain a new element

$$K \sharp_t L \in \mathcal{O}(\Sigma'', E''; \overline{A}_1, \dots, \overline{A}_{\overline{s}}, \overline{A}'_{t+1}, \dots, \overline{A}'_{\overline{s}'}; \underline{A}_1, \dots, \underline{A}_{\underline{s}-t}, \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'})$$

with the corresponding glued data. By Corollary 7, given orientations  $o_K$  and  $o_L$  of the determinant bundles of  $L$  and  $K$  we get an induced orientation  $o_K \sharp_t o_L$  of the determinant of  $K \sharp_t L$ . In addition this operation is associative.

(ii) Disjoint union of orientations. In addition to gluing we may form the disjoint union  $L_1 \amalg L_2$  of two operators  $L_1$  and  $L_2$ . There is a natural isomorphism from  $\text{Det}(L_1) \otimes \text{Det}(L_2) \rightarrow \text{Det}(L_1 \amalg L_2)$ . We denote by  $o_{L_1} \amalg o_{L_2}$  the induced orientation of  $\text{Det}(L_1 \amalg L_2)$ . This operation may look very trivial but it is not. In fact, it depends on the order of the components  $L_1$  and  $L_2$ .

Now we are ready to set up the necessary orientation algorithm.

**Step 0.** For each closed Riemann surface  $\Sigma$ , we can choose a  $\mathbb{C}$ -linear element  $L \in \mathcal{O}(\Sigma, E; \emptyset; \emptyset)$ , namely a  $\overline{\partial}$ -operator on  $E$ . The kernel and cokernel of this operator are complex vector spaces, and hence are canonically oriented. We define  $o_L$  to be this complex orientation.

**Step 1.** For each closed Reeb orbit  $\gamma \in \mathcal{P}_\alpha$ , there is a unique map  $\gamma : S^1 \rightarrow M$  parametrizing it with constant arc length such that  $\gamma(1)$  maps to the point  $z_\gamma$ . In addition we fix a symplectic trivialization of the normal bundle of  $\gamma$  (i.e. of  $\xi$  along the orbit). Thus we obtain a (periodic) family of symplectic matrices  $A_\gamma(t)$  by linearizing the Reeb flow. We pick the trivial symplectic vector bundle  $\theta^n$  of rank  $n$  over the Riemann sphere  $\mathbb{C}P^{1*}$  with one positive puncture identifying the restriction of  $\overline{\theta}$  to the boundary with  $\xi|_\gamma$  using

the trivialization. We fix an element  $\overline{L}_\gamma \in \mathcal{O}(\mathbb{CP}^1, \theta; A_\gamma; \emptyset)$  (note that these sets are all non-empty) together with an orientation  $\sigma_{\overline{L}_\gamma}$  of the determinant of the differential operator  $\overline{L}_\gamma$ .

**Step 2.** For each closed Reeb orbit  $\gamma \in \mathcal{P}_\alpha$  pick the trivial symplectic vector bundle  $\theta^n$  of rank  $n$  over the Riemann sphere  $\mathbb{CP}^{1*}$  with one negative puncture and fix an element  $\underline{L}_\gamma \in \mathcal{O}(\mathbb{CP}^1, \theta; \emptyset; A_\gamma)$ . We define the orientation  $o_{\underline{L}_\gamma}$  of its determinant so that  $o_{\underline{L}_\gamma} \sharp_1 o_{\overline{L}_\gamma}$  coincides with the complex orientation induced by the usual  $\overline{\partial}$ -operator on  $\mathbb{CP}^1$ .

**Step 3.** For  $L_i \in \mathcal{O}(\Sigma, E; A_i; \emptyset)$ ,  $i = 1, \dots, k$ , we define  $o_{L_1 \amalg \dots \amalg L_k} = o_{L_1} \amalg \dots \amalg o_{L_k}$ . We have a similar definition for negative punctures. For each

$$K \in \mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}}),$$

we define the orientation  $o_K$  of its determinant so that

$$o_{\underline{L}_{\overline{s}}} \amalg \dots \amalg \underline{L}_{\overline{1}} \sharp_{\overline{s}} o_K \sharp_{\underline{s}} o_{\overline{L}_{\underline{s}}} \amalg \dots \amalg \overline{L}_{\underline{1}}$$

coincides with the complex orientation induced by the usual  $\overline{\partial}$ -operator on  $\Sigma$ .

Note that step 3 is compatible with steps 1 and 2, i.e. the orientations  $\sigma_{\overline{L}_\gamma}$  and  $o_{\underline{L}_\gamma}$  obtained using step 3, coincide with the ones defined in step 1 and 2, respectively, for each closed Reeb orbit  $\gamma \in \mathcal{P}_\alpha$ .

Now we have to determine the behavior of these orientations under the gluing operation  $\sharp_t$ . Let us first describe the situation of two surfaces  $\Sigma$  and  $\Sigma'$  and operators  $K \in \mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{t}})$ ,  $L \in \mathcal{O}(\Sigma', E'; \overline{A}'_1, \dots, \overline{A}'_{\overline{t}}; \underline{A}'_1, \dots, \underline{A}'_{\underline{s}})$  with  $\underline{A}_m$  matching  $\overline{A}'_m$ , for  $m = 1, \dots, t$ . That means we are in a situation of a complete gluing, i.e.  $K \sharp_t L \in \mathcal{O}(\Sigma'', E''; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}'_1, \dots, \underline{A}'_{\underline{s}})$ .

**Proposition 8.** *In the situation described above we have*

$$o_K \sharp_t o_L = o_{K \sharp_t L}.$$

*Proof.* Note that the statement is true by definition if  $t = 1$ , when  $\Sigma = \mathbb{CP}^1$  and  $\overline{s} = 0$  or when  $\Sigma' = \mathbb{CP}^1$  and  $\underline{s}' = 0$ , since we simply cap off a puncture.

Next, note that the statement also holds if  $\overline{s} = 1$  or  $\underline{s}' = 1$  instead of zero, i.e. in the case of gluing a cylinder. Indeed, we can reduce this case to the previous one after capping off the other end of the cylinder, by associativity of the gluing operation  $\sharp$  for orientations.

Consider now the case of Riemann surfaces with an arbitrary topology and arbitrary number of punctures, equipped with operators  $K$  and  $L$  such that their asymptotic expression near the punctures are complex linear. The operators  $K$  and  $L$  are then homotopic to a complex linear operator in the class of operators with fixed asymptotics at the punctures. Hence, their determinants inherit a natural orientation from their complex

structure. The same is of course true for the glued operator and it is clear that gluing complex orientations yields the complex orientation since the gluing map is homotopic to a complex linear isomorphism in this case.

Finally, we reduce the general case to the above situation by gluing cylinders  $Z$  and  $Z'$  to the punctures at which we want to glue  $K$  and  $L$ . We choose  $Z$  and  $Z'$  so that the glued operators  $K \sharp Z$  and  $Z' \sharp L$  have complex linear asymptotic expressions at the negative and positive punctures, respectively. Since we can cap off all positive or negative ends of  $K$  and  $L$ , we may assume that there are no such punctures. We have a homotopy  $K \sharp L \cong K \sharp Z \sharp Z' \sharp L$  and hence

$$o_{K \sharp L} = o_{K \sharp Z \sharp Z' \sharp L} = o_{K \sharp Z} \sharp o_{Z' \sharp L}$$

due to consistency of gluing in the complex linear case. The latter is equal to

$$(o_{K \sharp Z}) \sharp (o_{Z' \sharp L}) = ((o_{K \sharp Z}) \sharp o_{Z'}) \sharp o_L = o_{K \sharp Z \sharp Z'} \sharp o_L = o_{K \sharp L}.$$

□

We are ready to derive the property for reordering the punctures.

*Proof of Theorem 2.* This property follows from the consistency of the orientations under disjoint union applied to the 1-punctured spheres used to cap off operators over arbitrary surfaces with punctures. Comparing the orientations via  $\bar{u}_{l,l+1}$  is the same as comparing the orientations on the disjoint unions  $\sigma_{\bar{L}_l} \amalg \sigma_{\bar{L}_{l+1}}$  and  $\sigma_{\bar{L}_{l+1}} \amalg \sigma_{\bar{L}_l}$  of  $\bar{L}_l \in \mathcal{O}(\mathbb{C}P^1, \theta; \bar{A}_{\underline{\gamma}_l}; \emptyset)$  and  $\bar{L}_{l+1} \in \mathcal{O}(\mathbb{C}P^1, \theta; \bar{A}_{\underline{\gamma}_{l+1}}; \emptyset)$ . This gives an orientation of the (virtual) vector space  $\ker(\bar{L}_l) \oplus \ker(\bar{L}_{l+1}) \ominus (\text{coker}(\bar{L}_l) \oplus \text{coker}(\bar{L}_{l+1}))$  which is given by orientations of the two vector spaces of the difference. Thus we can compare the orientations induced by either of the two possible orders of the components with that given one. It is not hard to check that this differs exactly if both indices  $\text{ind}(\bar{L}_l)$  and  $\text{ind}(\bar{L}_{l+1})$  are odd. But

$$\text{ind}(\bar{L}_l) = \mu(\underline{\gamma}_l) + (n-1).$$

The proof for the case of changing the ordering at the positive ends is almost the same; in that case, we have to use spheres with one negative puncture and the index is

$$\text{ind}(\underline{L}_k) = -(\mu(\bar{\gamma}_k) + (n-1)).$$

□

Next we discuss the behavior of coherent orientations under the disjoint union operation.

**Proposition 9.** *Let*

$$K \in \mathcal{O}(\Sigma, E; \bar{A}_1, \dots, \bar{A}_s; \underline{A}_1, \dots, \underline{A}_s)$$

and

$$L \in \mathcal{O}(\Sigma', E'; \overline{A}_1, \dots, \overline{A}_{\overline{s}'}; \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'}).$$

Then, for

$$K \amalg L \in \mathcal{O}(\Sigma \amalg \Sigma', E \amalg E'; \overline{A}_1, \dots, \overline{A}_{\overline{s}}, \overline{A}'_1, \dots, \overline{A}'_{\overline{s}'}; \underline{A}_1, \dots, \underline{A}_{\underline{s}}, \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'}),$$

we have  $o_{K \amalg L} = (-1)^\epsilon o_L \amalg o_K$ , where

$$\epsilon = \left( \sum_{l=1}^{\underline{s}} (\mu(\gamma_l) + (n-3)) \right) \left( \sum_{k=1}^{\overline{s}'} (\mu(\overline{\gamma}'_k) + (n-3)) \right).$$

*Proof.* By definition,

$$o_{\underline{L}_{\overline{\gamma}'_{\overline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}'_1}} \amalg o_{\underline{L}_{\overline{\gamma}_{\overline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}_1}} \amalg_{\overline{s}+\overline{s}'} o_K \amalg o_{\underline{L}_{\underline{s}+\underline{s}'}} o_{\underline{L}_{\underline{\gamma}'_{\underline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}'_1}} \amalg o_{\underline{L}_{\underline{\gamma}_{\underline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}_1}}$$

coincides with the complex orientation. On the other hand,

$$\begin{aligned} & o_{\underline{L}_{\overline{\gamma}'_{\overline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}'_1}} \amalg o_{\underline{L}_{\overline{\gamma}_{\overline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}_1}} \amalg_{\overline{s}+\overline{s}'} o_K \amalg o_{\underline{L}_{\underline{s}+\underline{s}'}} o_{\underline{L}_{\underline{\gamma}'_{\underline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}'_1}} \amalg o_{\underline{L}_{\underline{\gamma}_{\underline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}_1}} \\ &= o_{\underline{L}_{\overline{\gamma}'_{\overline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}'_1}} \amalg_{\overline{s}'} o_{\underline{L}_{\overline{\gamma}_{\overline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}_1}} \amalg_{\overline{s}} o_K \amalg o_{\underline{L}_{\underline{s}+\underline{s}'}} o_{\underline{L}_{\underline{\gamma}'_{\underline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}'_1}} \amalg_{\underline{s}} o_{\underline{L}_{\underline{\gamma}_{\underline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}_1}} \\ &= (-1)^\epsilon o_{\underline{L}_{\overline{\gamma}'_{\overline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}'_1}} \amalg_{\overline{s}'} o_{\underline{L}_{\underline{s}+\underline{s}'}} o_{\underline{L}_{\underline{\gamma}'_{\underline{s}'}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}'_1}} \amalg o_{\underline{L}_{\overline{\gamma}_{\overline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\overline{\gamma}_1}} \amalg_{\overline{s}} o_K \amalg_{\underline{s}} o_{\underline{L}_{\underline{\gamma}_{\underline{s}}}} \amalg \dots \amalg o_{\underline{L}_{\underline{\gamma}_1}} \end{aligned}$$

But the latter is a complex orientation multiplied by  $(-1)^\epsilon$ , by definition of  $o_K$  and  $o_L$ . Comparison with the definition of  $o_{K \amalg L}$  gives the desired result.  $\square$

The following statement about the behavior of coherent orientations for a general gluing is a mixed case of Propositions 8 and 9.

**Corollary 10.** *Let*

$$K \in \mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}})$$

and

$$L \in \mathcal{O}(\Sigma', E'; \overline{A}'_1, \dots, \overline{A}'_{\overline{s}'}; \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'}),$$

with  $\underline{A}_{\underline{s}+1-m}$  matching  $\overline{A}'_m$  for  $m = 1, \dots, t \leq \min\{\underline{s}, \overline{s}'\}$ . Then, for

$$K \amalg_t L \in \mathcal{O}(\Sigma \amalg_t \Sigma', E \amalg_t E'; \overline{A}_1, \dots, \overline{A}_{\overline{s}}, \overline{A}'_{t+1}, \dots, \overline{A}'_{\overline{s}'}; \underline{A}_1, \dots, \underline{A}_{\underline{s}-t}, \underline{A}'_1, \dots, \underline{A}'_{\underline{s}'}),$$

we have  $o_{K \amalg_t L} = (-1)^\epsilon o_K \amalg_t o_L$ , where

$$\epsilon = \left( \sum_{l=1}^{\underline{s}-t} (\mu(\gamma_l) + (n-3)) \right) \left( \sum_{k=t+1}^{\overline{s}'} (\mu(\overline{\gamma}'_k) + (n-3)) \right).$$

*Proof.* By definition,

$$o_{L_{\overline{\gamma}_{s'}}} \amalg \dots \amalg L_{\overline{\gamma}_{t+1}} \amalg L_{\overline{\gamma}_s} \amalg \dots \amalg L_{\overline{\gamma}_1} \#_{\overline{s}+\overline{s}'-t} o_K \#_t L \#_{\underline{s}+\underline{s}'-t} o_{L_{\underline{\gamma}'_s}} \amalg \dots \amalg L_{\underline{\gamma}'_1} \amalg L_{\underline{\gamma}_{s-t}} \amalg \dots \amalg L'_{\underline{\gamma}_1}$$

coincides with the complex orientation. On the other hand,

$$o_{L_{\overline{\gamma}'_s}} \amalg \dots \amalg L_{\overline{\gamma}'_{t+1}} \amalg L_{\overline{\gamma}'_s} \amalg \dots \amalg L_{\overline{\gamma}'_1} \#_{\overline{s}+\overline{s}'-t} (o_K \#_t o_L) \#_{\underline{s}+\underline{s}'-t} o_{L_{\underline{\gamma}'_s}} \amalg \dots \amalg L_{\underline{\gamma}'_1} \amalg L_{\underline{\gamma}_{s-t}} \amalg \dots \amalg L'_{\underline{\gamma}_1}$$

coincides with

$$(o_{L_{\overline{\gamma}'_s}} \amalg \dots \amalg L_{\overline{\gamma}'_{t+1}} \amalg o_{L_{\overline{\gamma}_s}} \amalg \dots \amalg L_{\overline{\gamma}_1} \#_{\overline{s}K}) \#_{\underline{s}+\underline{s}'-t} (o_{L_{\underline{\gamma}'_s}} \amalg \dots \amalg L_{\underline{\gamma}'_1} \amalg o_{L_{\underline{\gamma}_{s-t}}} \amalg \dots \amalg L_{\underline{\gamma}_1})$$

because of associativity. In order to apply Proposition 8 to the gluing  $\#_{\underline{s}+\underline{s}'-t}$ , we have to permute several punctures in the right term :  $\overline{A}'_{t+1}, \dots, \overline{A}'_{s'}$  must be exchanged with  $\underline{A}_{s-t}, \dots, \underline{A}_1$ . This gives precisely the sign  $(-1)^\epsilon$ . After the gluing  $\#_{\underline{s}+\underline{s}'-t}$ , we obtain a complex orientation, and comparison with the definition of  $o_{L \#_t K}$  gives the desired result.  $\square$

Finally, we study the behavior of coherent orientations under the action of automorphisms.

**Proposition 11.** *Let  $L \in \mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_s; \underline{A}_1, \dots, \underline{A}_s)$  and  $\sigma$  be a diffeomorphism of  $\Sigma$  preserving the punctures. Let  $j' = \sigma_* j$  and  $K = \sigma(L)$ . Then the map induced by  $\sigma$  from the determinant line of  $L$  to the determinant line of  $K$  preserves coherent orientations.*

*Proof.* First, if  $\Sigma$  is a closed Riemann surface, we can deform  $L$  into a  $\mathbb{C}$ -linear operator  $L'$  by a path of operators. The isomorphism induced by  $\sigma$  from the kernel and cokernel of  $L'$  to the kernel and cokernel of  $\sigma(L')$  is  $\mathbb{C}$ -linear, so it sends  $o_{L'}$  to  $o_{\sigma(L')}$  and hence  $o_L$  to  $o_K$ . In the general case, consider the action of  $\sigma$  on  $o_{L_{\overline{\gamma}_s}} \amalg \dots \amalg L_{\overline{\gamma}_1} \#_{\overline{s}} o_L \#_{\underline{s}} o_{L_{\underline{\gamma}_s}} \amalg \dots \amalg L_{\underline{\gamma}_1}$ . On one hand,  $\sigma$  preserves the complex orientation as above. On the other hand, the action of  $\sigma$  clearly commutes with the gluing maps, and by assumption  $\sigma$  extends as the identity on each capping 1-punctured sphere. Therefore the coherent orientations must be preserved as well.  $\square$

This proposition shows that the determinant line bundle for the Fredholm operators remains trivial if we allow the conformal structure on  $\Sigma$  to vary. Moreover, for  $j' = j$  and  $K = L$ , it shows that the coherent orientations descend to the quotient space, since they are preserved by the automorphisms.

**Remark 12.** *One is tempted to consider the space of Fredholm operators with asymptotics in a connected component of  $\mathcal{S}$  instead of concrete descriptions. Unfortunately, this is not possible, due to Theorem 2 in [3]. Neither of the determinant bundles over such spaces is orientable !*



**Remark 13.** *We can compare the coherent orientations constructed in this paper with the orientations introduced in section 1.8 of [2]. Even though our definition of coherent orientations is formulated differently than axioms (C1)-(C3) of [2], the 2 notions actually coincide, provided we choose the  $1 \otimes 1^*$  orientation for each cylinder equipped with a translation invariant operator  $L \in \mathcal{O}(\mathbb{CP}^1, \theta; A_\gamma; A_\gamma)$ .*

#### 4. MODULI SPACES OF $J$ -HOLOMORPHIC CURVES

We will apply our construction of coherent orientations to moduli spaces of holomorphic curves in a symplectic cobordism  $(W^{2n}, \omega)$  of dimension  $2n$ . We describe the Banach setting in which we study holomorphic curves in symplectic manifolds with ends and show that the linearization of the  $\bar{\partial}$ -equation gives rise to a Fredholm operator between Banach spaces. The approach we finally chose was explained to us by Helmut Hofer.

As usual we denote by  $\mathcal{T}_{g,s}$ , the Teichmüller space of genus  $g$  smooth complex curves with  $s$  distinct marked points. This is a smooth complex manifold of complex dimension  $3g - 3 + s$ . We represent  $\mathcal{T}_{g,s}$  by a family of conformal structures on a (fixed) closed oriented surface  $\Sigma$  and  $s$  distinct points. We therefore denote by  $(j, x_1, \dots, x_s) \in \mathcal{T}_{g,s}$  the corresponding data on  $\Sigma$ .

Let  $p > 2$  and  $k \geq 1$ . We denote by

$$\mathcal{B}_{W,J}^\Sigma(\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{s}}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}})$$

the set of elements consisting of  $(j, \bar{x}_1, \dots, \bar{x}_{\bar{s}}, \underline{x}_1, \dots, \underline{x}_{\underline{s}}) \in \mathcal{T}_{g, \bar{s} + \underline{s}}$  and a map  $u : \Sigma \setminus \{\bar{x}_1, \dots, \bar{x}_{\bar{s}}, \underline{x}_1, \dots, \underline{x}_{\underline{s}}\} \rightarrow W$  which is locally in  $L_k^p$ . For each puncture there is a neighborhood which is completely mapped inside one of the ends of  $W$ : for a positive puncture this will be the positive end, for a negative puncture it will be the negative end. Furthermore, we ask that the projection of the map to the contact manifold corresponding to such an end maps a small neighborhood of  $\bar{x}_k$  or  $\underline{x}_l$  completely inside a fixed tubular neighborhood of the closed Reeb orbit  $\bar{\gamma}_k \in \mathcal{P}_{\alpha_+}$  or  $\underline{\gamma}_l \in \mathcal{P}_{\alpha_-}$ . We introduce coordinates  $(\vartheta, \zeta) \in S^1 \times D^{2n-2}$  in a tubular neighborhood of the geometric locus  $|\gamma|$  of a closed  $\alpha_\pm$ -Reeb orbit  $\gamma$  with multiplicity  $m(\gamma)$ , such that  $\{0\} \times S^1 = |\gamma|$  parameterizes the underlying simple closed Reeb orbit with constant arc length element. We require that  $\vartheta = 0$  corresponds to the fixed point on  $\gamma$ .

Finally, near a puncture we require that there are holomorphic cylindrical coordinates  $(s, \theta) \in [R_0, \infty) \times S^1$  for a positive puncture and  $(s, \theta) \in (-\infty, -R_0] \times S^1$  for a negative puncture, such that the components of the map  $u = (u_{\mathbb{R}}, u_M)$ , with  $u_M = (\vartheta, \zeta)$ , satisfy  $(u_{\mathbb{R}} - \mathcal{A}(\gamma)s - a_0), (\vartheta - m(\gamma)\theta), \zeta \in L_k^p(e^{d|s|} ds d\theta)$  in the tubular coordinates around the corresponding Reeb orbit.

Here  $\mathcal{A}(\gamma) = \int_{\gamma} \alpha_{\pm}$  is the action of the (multiple) closed Reeb orbit  $\gamma$ ,  $a_0$  is a constant,  $0 < d/p < 1$  is smaller than the absolute value of the biggest negative eigenvalues of  $L_{S_{\gamma_k}}^{\infty}$  for  $\bar{x}_k$  or smaller than the smallest positive eigenvalue of  $L_{S_{\underline{x}_l}}^{\infty}$  for  $\underline{x}_l$ , and  $L_{S_{\gamma}}^{\infty}$  is an operator acting on smooth functions in the following way : fix a hermitian trivialization of  $\xi$  over  $\gamma$ , i.e. trivialize it as a complex and symplectic bundle. Then  $L_{S_{\gamma}}^{\infty} := -i\frac{\partial}{\partial\theta} - S_{\gamma}(\theta)$  where  $S_{\gamma}(\theta) = -i\pi_{m(\gamma)\theta}dR(m(\gamma)\theta)\pi_{m(\gamma)\theta}$  is a family of symmetric matrices,  $\pi : TM \longrightarrow \xi$  is the orthogonal projection (see [5]). Obviously, maps in  $\mathcal{B}_{W,J}^{\Sigma}(\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{s}}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}})$  are asymptotic to  $\bar{\gamma}_k$  in  $\bar{x}_k$  at  $+\infty$  and to  $\underline{\gamma}_l$  in  $\underline{x}_l$  at  $-\infty$ .

Remember that the asymptotics impose a condition on the choice of complex coordinates near the punctures. Namely, there is a finite set of  $m(\gamma)$  possible *directions*  $\frac{\partial}{\partial x}$  at the punctures, since  $\lim_{x \rightarrow 0} u_M(x) = z_{\gamma}$  for  $x \in \mathbb{R}$ , due to our choices. Finally, we consider as an element in  $\mathcal{B}$  the map  $u$  together with a choice of possible directions at each puncture. We may think of the additionally introduced direction at the punctures as a puncture on the boundary of the compactification of the Riemann surface to a surface with boundary using the limits.

In [5] the authors show that punctured holomorphic curves with *finite energy* which appear naturally in SFT are asymptotic to Reeb orbits at the punctures and satisfy the above decay conditions such that they lie in the set  $\mathcal{B}$ . Moreover, with its obvious topology this set becomes a Banach manifold. This can be seen using the exponential map on  $W$ , corresponding to a complete Riemannian metric which is cylindrical on the ends of  $W$ .

Let  $\mathcal{Hol}_{W,J}^{\Sigma}(\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{s}}, \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}})$  be the set of punctured holomorphic maps with the given asymptotics. The above discussion shows that  $\mathcal{Hol}_{W,J}^{\Sigma} \subset \mathcal{B}_{W,J}^{\Sigma}$ , so that  $\mathcal{B}$  is an appropriate configuration space to contain the set of holomorphic maps. Notice that in order to obtain the moduli space  $\mathcal{M}_{W,J}^{\Sigma}(\bar{\gamma}_1, \dots, \bar{\gamma}_{\bar{s}}, \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}})$  of punctured holomorphic curves we still need to divide out by the mapping class group  $\text{Diff}(\Sigma, x)/\text{Diff}_0(\Sigma, x)$ .

The tangent space  $T_{(j,x,u)}\mathcal{B}$  splits into the tangent space of  $\mathcal{T}_{g,\bar{s}+\underline{s}}$  at  $(j, x)$  and the vector space of sections of  $u^*TW$  which are locally  $L_k^p$  and, near the punctures, are either  $L_k^{p,d}(TW)$  in cylindrical coordinates or are constant linear combinations of  $\frac{\partial}{\partial t}$  and  $R_{\alpha_{\pm}}$ .

At each point  $(j, x, u) \in \mathcal{B}$  we have a Banach space

$$E_{(j,x,u)} := L_{k-1}^{p,d}(\Lambda^{0,1}(u^*TW))$$

forming a smooth Banach bundle  $\mathcal{E} \longrightarrow \mathcal{B}$ . Then the equation for  $J$ -holomorphic curves defines a Fredholm section  $\bar{\partial} : \mathcal{B} \longrightarrow \mathcal{E}$ .

Consider the linearization  $\bar{\partial}_{(j,x,u)}$  of this section at  $(j, x, u) \in \mathcal{B}$ . This operator splits into

$$\bar{\partial}_{(j,x,u)} : D_{(j,x,u)} \oplus L_{(j,x,u)} : T_{(j,x)}\mathcal{T}_{g,\bar{s}+\underline{s}} \oplus \Gamma_{\bar{s},\underline{s}} \oplus L_k^{p,d}(u^*TW) \longrightarrow L_{k-1}^{p,d}(\Lambda^{0,1}(u^*TW)).$$

It follows from the above discussion that the second summand  $L_{(j,x,u)}$  is an element of  $\mathcal{O}(\Sigma, E; \overline{A}_1, \dots, \overline{A}_{\overline{s}}; \underline{A}_1, \dots, \underline{A}_{\underline{s}})$ .

Therefore, by Proposition 4,  $L_{(j,x,u)}$  is Fredholm and has index

$$\text{ind}(L_{(j,x,u)}) = \sum_{k=1}^{\overline{s}} (\mu(\overline{\gamma}_k) - (n-1)) - \sum_{l=1}^{\underline{s}} (\mu(\underline{\gamma}_l) + (n-1)) + n\chi(\Sigma) + 2c_1(W, \omega)[A].$$

Since the first summand  $D_{(j,x,u)}$  of  $\overline{\partial}_u$  has finite rank, from the index above one can easily derive the expected dimension of the moduli spaces :

$$\sum_{k=1}^{\overline{s}} \mu(\overline{\gamma}_k) - \sum_{l=1}^{\underline{s}} \mu(\underline{\gamma}_l) + (n-3)(\chi(\Sigma) - \overline{s} - \underline{s}) + 2c_1(W, \omega)[A].$$

*Proof of Theorem 1.* First, we have to explain how to orient the determinant bundles of the moduli spaces  $\mathcal{M}_{W,J}^{A,\Sigma}(\overline{\gamma}_1, \dots, \overline{\gamma}_{\overline{s}}, \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}})$  using the construction of the previous section.

Let  $(j, x, u) \in \mathcal{Hol}_{W,J}^{A,\Sigma}(\overline{\gamma}_1, \dots, \overline{\gamma}_{\overline{s}}, \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}})$ . Consider the linearized operator  $\overline{\partial}_{(j,x,u)}$  splitting as  $D_{(j,x,u)} \oplus L_{(j,x,u)}$ . The family of operators  $L_{(j,x,u)}$  for  $(j, x, u) \in \mathcal{B}$  defines a continuous map

$$\begin{aligned} op : \mathcal{Hol}_{W,J}^{A,\Sigma}(\overline{\gamma}_1, \dots, \overline{\gamma}_{\overline{s}}, \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}}) &\longrightarrow \mathcal{O}(\Sigma, u^*TW; A_{\overline{\gamma}_1}, \dots, A_{\overline{\gamma}_{\overline{s}}}; A_{\underline{\gamma}_1}, \dots, A_{\underline{\gamma}_{\underline{s}}}) \\ (j, x, u) &\longrightarrow L_{(j,x,u)}. \end{aligned}$$

The operator  $\overline{\partial}_{(j,x,u)}$  is homotopic to  $0 \oplus L_{(j,x,u)}$ , a stabilization of the operator  $L_{(j,x,u)} \in \mathcal{O}$  with the complex vector space  $T_{(j,x)}\mathcal{T}_{g,\overline{s}+\underline{s}}$ . Hence, their determinant spaces are canonically isomorphic and the determinant bundle over  $\mathcal{Hol}$  corresponding to the Fredholm operator  $\overline{\partial}_{(j,x,u)}$  is isomorphic to  $op^*\text{Det}(\mathcal{O})$ . In particular, the orientations from Section 3 induce orientations on the determinant bundle of  $\mathcal{Hol}$ .

The mapping class group of  $(\Sigma, x)$  acts naturally on  $\mathcal{Hol}$ . Let  $\mathcal{M}$  be the moduli space of this action. By Proposition 11, the coherent orientations are preserved by this action and descend to the moduli spaces  $\mathcal{M}$ .

Next, we have to show that, given compact subsets

$$K_1 \subset \mathcal{M}_{W_1,J_1}^{\Sigma_1}(\overline{\gamma}_1, \dots, \overline{\gamma}_{\overline{s}}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}-t}, \beta_1, \dots, \beta_t)$$

and

$$K_2 \subset \mathcal{M}_{W_2,J_2}^{\Sigma_2}(\beta_t, \dots, \beta_2, \beta_1, \overline{\gamma}'_{t+1}, \dots, \overline{\gamma}'_{\overline{s}'}; \underline{\gamma}'_1, \dots, \underline{\gamma}'_{\underline{s}'}),$$

the gluing maps

$$\Phi_R : K_1 \times K_2 \longrightarrow \mathcal{M}_{W_R,J_R}^{\Sigma}(\overline{\gamma}_1, \dots, \overline{\gamma}_{\overline{s}}, \overline{\gamma}'_{t+1}, \dots, \overline{\gamma}'_{\overline{s}'}; \underline{\gamma}_1, \dots, \underline{\gamma}_{\underline{s}-t}, \underline{\gamma}'_1, \dots, \underline{\gamma}'_{\underline{s}'})$$

are orientation preserving up to a sign  $(-1)^\epsilon$  where  $\epsilon$  is determined by Corollary 10.

In order to prove this, we have to relate the differential of  $\Phi_R$  to the linear gluing map of Corollary 6. Note that the construction of the gluing map requires the Fredholm section  $\bar{\partial} : \mathcal{B} \rightarrow \mathcal{E}$  to be transverse to the zero section. In other words, the linearization  $\bar{\partial}_{(j,x,u)}$  of this section at  $(j, x, u) \in \bar{\partial}^{-1}(0) \subset \mathcal{B}$  must be a surjective operator. This transversality can be achieved in different ways, such as perturbing the almost complex structure  $J$  or perturbing the right hand side of the Cauchy-Riemann equation with an element of  $L_{k-1}^{p,d}(\Lambda^{0,1}(u^*TW))$  having compact support in the complementary of the punctures. In all cases, the linearization of the perturbed section  $\bar{\partial}$  is still an element of  $\mathcal{O}$  modulo some zero order terms not affecting our constructions. Therefore, the definition of coherent orientations is independent of the precise way we achieve transversality for the moduli spaces.

Let  $(j_i, x_i, u_i) \in K_i$  for  $i = 1, 2$ . Using cutoff functions near the closed orbits  $\beta_1, \dots, \beta_t$ , we construct a pre-glued map  $u_R$  into the glued cobordism  $(W_R, J_R)$ . Gluing the two conformal structures we also construct conformal data  $(j_R, x_R)$  on the glued surface  $\Sigma_1 \sharp_t \Sigma_2$ . Then the glued data  $(j_R, x_R, u_R)$  satisfy

$$\|\bar{\partial}u_R\| \leq C(R),$$

where  $C(R) \rightarrow 0$  as  $R \rightarrow \infty$ .

Under the above transversality assumption, the linearized operator  $\bar{\partial}_{(j_R, x_R, u_R)}$  is surjective and has a uniformly bounded right inverse, as in Proposition 5. An actual holomorphic curve  $(j_R, x_R, w_R)$  is obtained using Newton iterations (see [7]), in a neighborhood of  $(j_R, x_R, u_R)$  of size controlled by  $C(R)$ . In particular, the difference in norm of the linearizations  $\bar{\partial}_{(j_R, x_R, w_R)}$  and  $\bar{\partial}_{(j_R, x_R, u_R)}$ , and the glued operator  $\bar{\partial}_R$  obtained from  $\bar{\partial}_{(j_i, x_i, u_i)}$ , for  $i = 1, 2$  as in Section 2, are arbitrarily small for  $R$  sufficiently large.

The differential of the gluing map  $\Phi_R$  is the composition of the linearizations of the pre-gluing map and of the Newton iteration map. The linearization of the pre-gluing map involves gluing sections of  $L_k^{p,d}(u_i^*TW)$  for  $i = 1, 2$  using some cut-off functions. It is therefore of the same form as the linear gluing map  $g_R$  from Section 2. On the other hand, the differential of the Newton iteration map approaches the projection  $I - Q_{u_R}\bar{\partial}_{u_R}$  to the kernel of  $\bar{\partial}_{u_R}$  along the image of its right inverse  $Q_{u_R}$  as  $R$  becomes large. The above discussion shows that this projection is very close to the projection  $I - Q_R\bar{\partial}_R$  of Section 2 for  $R$  large.

This shows that the differential of  $\Phi_R$  and the linear gluing map are very close for  $R$  large, so that they induce the same map on orientations.  $\square$

Note that multiple covers of holomorphic curves have to be considered in Symplectic Field Theory. These prevent, even for a generic choice of  $J$ , the linearized Cauchy-Riemann operator from being surjective. Therefore, it is necessary to use more sophisticated methods, such as multi-valued perturbations [6], in order to achieve transversality. The moduli spaces we obtain in this way are *branched labeled pseudo-manifolds*. These are topological spaces consisting of finitely many smooth open strata of finite dimension such that the top-dimensional strata are dense. There is an assignment of positive rational numbers to each top-stratum. This discussion shows that, to define Symplectic Field Theory for all contact manifolds, we have to leave the realm of integer invariants, and rather use rational coefficients.

As soon as we wish to compute integral or rational invariants we have to define orientations of the moduli spaces of holomorphic curves which satisfy some coherence property under the gluing operation. This will lead to signs for holomorphic curves of index 1 in symplectizations (i.e. those used to define the differential) and for holomorphic curves of index 0 or  $-1$  in symplectic cobordisms (i.e. those which define homomorphisms and homotopies between the differential algebras). Counting them with their signs we get integers or rational numbers.

On the other hand, there could be situations where one is able to define  $\mathbb{Z}_2$ -contact homology (see for example [11]), and hence avoid coherent orientations. However, it is not obvious that one can justify the invariance even in the absence of holomorphic curves of index 1 : multiply covered cylinders occur quite naturally in the symplectic cobordism defining the chain map which should be chain homotopic to the identity. In that case, multi-valued perturbations could not be avoided, and rational coefficients would make coherent orientations necessary.

## 5. EVEN AND ODD BEHAVIOR OF THE ORIENTATION FOR MULTIPLY-COVERED ORBITS

We now establish the behavior of the orientations under change of asymptotic directions at punctures which are asymptotic to Reeb orbits with multiplicity.

*Proof of Theorem 3.* First note that, in view of steps 2 and 3 in Section 3 for the construction of coherent orientations, it is enough to prove the theorem for the moduli space  $\mathcal{M}^\Sigma(\gamma_m, \emptyset)$ , where  $\gamma_m$  is the  $m$ -fold covering of the simple orbit  $\gamma$ .

The index of the operator  $\overline{L}_\gamma$  is given by  $\mu(\gamma) + (n - 1)$ . Then  $\overline{L}_{\gamma_m}$  is the pullback of  $\overline{L}_\gamma$  under the branched covering  $z \mapsto z^m$ . In particular, this operator has a  $\mathbb{Z}_m$ -symmetry of rotations. Hence,  $\mathbb{Z}_m$  acts on the kernel and cokernel of  $\overline{L}_{\gamma_m}$ , and these vector spaces split into irreducible representations over  $\mathbb{R}$ .

If  $m$  is odd, there are no elements of even order, so the action of  $\mathbb{Z}_m$  preserves orientations. Hence, coherent orientations are invariant under a change of asymptotic direction.

If  $m$  is even, the possible irreducible representations include 2 representations of dimension 1 (trivial and sign change) and  $m - 2$  representations of dimension 2, generated by rotations of angle  $\frac{2\pi}{m}i$ ,  $i = 1, \dots, m - 1$ ,  $i \neq \frac{m}{2}$ . The trivial representations correspond to the kernel and cokernel of  $\overline{L}_\gamma$ . Therefore, the index difference

$$\text{ind}(\overline{L}_{\gamma_m}) - \text{ind}(\overline{L}_\gamma) = \mu(\gamma_m) - \mu(\gamma)$$

has the same parity as the multiplicity of the orientation reversing representation over  $\mathbb{R}$  in the kernel and cokernel of  $\overline{L}_{\gamma_m}$ . Hence, rotating the asymptotic direction of  $\frac{2\pi}{m}$  reverses the coherent orientation if and only if  $\mu(\gamma_m) - \mu(\gamma)$  is odd. This is exactly the case described in the theorem (see e.g. Lemma 3.2.4. in [10]).  $\square$

Coherent orientations can be constructed after we make some orientation choices for all closed Reeb orbits. We now explain how to reduce the number of necessary choices in the case of multiply covered orbits.

**Proposition 14.** *Let  $\gamma$  be a simple closed Reeb orbit. Fix an orientation of the determinant bundle over  $\mathcal{O}(\mathbb{CP}^1, \theta^n; A_\gamma, \emptyset)$ .*

- (i) *If  $\gamma'$  is an odd multiple of  $\gamma$ , there is a canonical choice for the orientation of the determinant bundle over  $\mathcal{O}(\mathbb{CP}^1, \theta^n; A_{\gamma'}, \emptyset)$  induced by the orientation for  $\gamma$ .*
- (ii) *The choice of an orientation for the double orbit  $\gamma'$ , in addition to  $\gamma$ , induces orientations for all multiple orbits.*
- (iii) *If  $n = 2$ , the orientation for  $\gamma$  alone induces orientations for even multiples as well, if they are good.*

*Proof.* (i) Let  $m$  be the odd multiplicity of  $\gamma'$ . Recall that  $\mathbb{Z}_m$  acts by rotation on the kernel and cokernel of  $\overline{L}_{\gamma'}$  and that these vector spaces split into the invariant part and some irreducible representations over  $\mathbb{R}^2$  generated by a rotation of angle  $\frac{2\pi}{m}i$ ,  $i = 1, \dots, m - 1$ . The invariant part consists of the pullback of the kernel and cokernel of the operator  $\overline{L}_\gamma$  and hence it is oriented. We choose the orientation on each irreducible representation  $\mathbb{R}^2$  in such a way that  $i < \frac{m}{2}$ .

(ii) If the multiplicity  $m$  of  $\gamma'$  is even, the kernel and cokernel of  $\overline{L}_{\gamma'}$  split into the invariant part, some irreducible representations over  $\mathbb{R}^2$  generated by a rotation of angle  $\frac{2\pi}{m}i$ ,  $i = 1, \dots, m - 1$ ,  $i \neq \frac{m}{2}$ , and the part  $V_{-1}$  on which the generator of  $\mathbb{Z}_m$  acts by  $-1$ . The invariant part and the  $\mathbb{R}^2$  summands are oriented as above. On the other hand, the kernel and cokernel of the operator corresponding to the double orbit split precisely into the invariant part and  $V_{-1}$ . Since the choice of an orientation for the simple orbit

is equivalent to an orientation of the invariant part, the choice of an orientation for the double orbit is just what we need to orient all the other multiples.

(iii) In the case  $n = 2$  we have to find a way to orient the part  $V_{-1}$  on which the rotation acts by  $-1$ . Choose the operator  $\overline{L}_\gamma$  so that it splits into the standard Cauchy-Riemann operator  $\overline{\partial}$  and an operator  $\overline{\partial}_{S_\gamma}$  of the form

$$\frac{\partial}{\partial s} + i \frac{\partial}{\partial t} + S_\gamma(t)$$

near the positive puncture. Since  $\overline{\partial}_{S_\gamma}$  has nondegenerate asymptotics, its Fredholm index is given by  $\mu(S_\gamma) - 1$ . This kind of operators were well studied in [4]. We now summarize some facts about  $\overline{\partial}_{S_\gamma}$  from that paper, assuming  $\text{ind}(\overline{\partial}_{S_\gamma}) \geq 0$ . Otherwise, we consider its formal adjoint, exchanging kernel and cokernel.

- a) Let  $\phi_\lambda$  be an eigenvector with eigenvalue  $\lambda \in \mathbb{R}$  of the operator  $L_{S_\gamma}^\infty := i \frac{\partial}{\partial t} + S_\gamma(t)$ . It has a well-defined winding number  $w(\phi_\lambda)$  in the fixed trivialization. Each winding number appears exactly twice and increases with the eigenvalue. Two eigenvectors with the same winding number are *pointwise* linearly independent in  $\mathbb{R}^2$  or a multiple of each other.
- b) The operator  $\overline{\partial}_{S_\gamma}$  is surjective. Elements in  $\ker \overline{\partial}_{S_\gamma}$  have the form  $Ce^{s\lambda}\phi_\lambda + O(e^{s\lambda})$ , where  $C \neq 0$ ,  $\lambda < 0$  and  $1 \leq w(\phi_\lambda) \leq w(S_\gamma)$ , where

$$w(S_\gamma) = \max\{w(\phi_\lambda) \mid 0 > \lambda \in \text{spec}(i \frac{\partial}{\partial t} + S_\gamma(t))\}.$$

- c) Conersely, we may choose a basis of  $\ker \overline{\partial}_{S_\gamma}$  such that the leading terms are in  $1-1$  correspondence to the set of eigenvectors  $\{\phi_\lambda\}_\lambda$  satisfying the above conditions.

Therefore, the Maslov index of  $S_\gamma$  can be computed in terms of these data. We have

$$\mu(S_\gamma) = 2w(S_\gamma) + p(S_\gamma)$$

where  $p(S_\gamma) = 0$  if the winding of the smallest positive eigenvalue is the same as that of the biggest negative, and  $p(S_\gamma) = 1$  otherwise.

Let  $\gamma'$  be the double cover of the orbit  $\gamma$ . Let  $\overline{L}_{\gamma'}$  be the pullback of the operator  $\overline{L}_\gamma$  under the branched covering  $z \mapsto z^2$ . The kernel of  $\overline{L}_{\gamma'}$  splits into the pullback of  $\ker \overline{L}_\gamma$  and the part  $V_{-1}$  that is not invariant under the  $\mathbb{Z}_2$  action. By assumption,  $\gamma'$  is good, so that  $p(S_{\gamma'}) = p(S_\gamma)$ . Therefore,  $V_{-1}$  splits into summands  $\mathbb{R}^2$  generated by 2 eigenvectors of equal winding number. Since these 2 eigenvectors are pointwise linearly independent, the corresponding summand  $\mathbb{R}^2$  inherits a natural orientation from the complex orientation of  $\xi$ . Hence, we obtain a natural orientation on  $V_{-1}$ .  $\square$

**Remark 15.** *If we reduce the number of choices in the construction of coherent orientations using Proposition 14, we can extract more precise information from the moduli spaces. Consider the symplectic cobordism  $(\mathbb{R} \times M, d(e^t \alpha_s))$ , where  $\alpha_s$  interpolates between contact forms  $\alpha_0$  and  $\alpha_1$  for isomorphic contact structures  $\xi_0 \cong \xi_1$ . Counting holomorphic curves of index 0 in this cobordism, we obtain an isomorphism  $\Phi$  between the contact homologies computed using  $\alpha_0$  and  $\alpha_1$ . Assume the image of a generator  $\gamma$  has the form  $\Phi(\gamma) = c\tilde{\gamma}$ , where  $c \neq 0$ . If we choose orientations separately for each closed orbit, then of course the sign of  $c$  is meaningless. However, if we choose orientations using Proposition 14, then the sign of  $c$  may contain some extra information.*

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FRÉDÉRIC BOURGEOIS, CENTRE DE MATHÉMATIQUES, ECOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE

*E-mail address:* fbourgeo@math.polytechnique.fr

KLAUS MOHNKE, INSTITUT FÜR MATHEMATIK, HUMBOLDT-UNIVERSITÄT ZU BERLIN, UNTER DEN LINDEN 6, D-10099 BERLIN, GERMANY

*E-mail address:* mohnke@mathematik.hu-berlin.de